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HEAT TRANSFER IN STRAIGHT CHANNELS WITH TWO-DIMENSIONAL CROSS-SECTIONAL PROFILES UNDER CONDITIONS OF NONISOTHERMAL STABILIZED FLOW OF A FLUID

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Heat transfer in stabilized regimes of nonisothermal flows in a round tube and in channels with two-dimensional convex cross sections is investigated with allowance for the velocity profile of a heating or cooling flow of a fluid. Mathematical models of unsteady processes of cooling by convection and radiation in pulling of rods and plane sheets at a constant velocity and of heat transfer (mass transfer) in a thin running-down layer of a medium (fluid) are solved.

Theoretical studies of unsteady heat transfer in channels with convex cross sections (an ellipse, a rhombus, an isosceles triangle, a sector of a circle, etc.) which are symmetric relative to the x-axis, in relative variables

$$\xi = \frac{x}{h}\,, \quad \eta = \frac{y}{b}\,, \quad \text{Fo} = \frac{at}{h^2}\,, \quad X = \frac{1}{\text{Pe}}\frac{z}{h}\,, \quad \text{Pe} = \frac{w_0 h}{a}\,, \quad 0 \le X < \infty\,, \quad 0 \le \text{Fo} < \infty\,,$$

are related to a solution of the equation

$$\frac{\partial T}{\partial \text{Fo}} + w(\xi, \eta)\frac{\partial T}{\partial X} = \frac{\partial^2 T}{\partial \xi^2} + \beta\frac{\partial^2 T}{\partial \eta^2} + \frac{q_v\,\psi_0(\xi, \eta)\,h^2}{\lambda} f(X, \text{Fo})\,, \quad \beta = \frac{h^2}{b^2} \tag{1}$$

with different internal and external heat loadings, for instance, with the boundary conditions of the first kind

$$[T(\xi, \eta, X, \text{Fo})]_{X=0} = \varphi_0(\text{Fo})\,, \quad [T(\xi, \eta, X, \text{Fo})]_{\text{Fo}=0} = T_0\,, \quad [T(\xi, \eta, X, \text{Fo})]_{\Gamma} = \varphi(\xi, X, \text{Fo})\,, \tag{2}$$

where Γ is the interior lateral surface of a thermally thin channel wall.

The numerical-analysis solving algorithm of [1] of the simultaneous use of the Laplace–Carson double integral transformation [2] with respect to unilateral parabolic variables X and Fo and the method of finite elements with implementation of the orthogonal projection of the discrepancy within the entire range of bilateral elliptic coordinates ξ, η in the variety of the representation of the solution

$$\overline{T}_n^*(\xi, \eta, s, p) = \overline{\varphi}^*(\xi, s, p) + \sum_{k=1}^{n} \overline{a}_k^*(s, p)\,\psi_k(\xi, \eta) \tag{3}$$

leads relative to the matrix-response $\|\overline{a}^*(s, p)\|$ to the equation

$$\|A + sB + pC\|\;\|\overline{a}^*(s, p)\| = [T_0 - \overline{\varphi}^*(s, p)]\,p\,\|N\| + [\overline{\varphi}_0(p) - \overline{\varphi}^*(s, p)]\,s\,\|F\| + \frac{q_v h^2}{\lambda}\overline{f}^*(s, p)\,\|E\|\,, \tag{4}$$

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wherein, without loss of generality of the method, it is assumed that $\overline{\varphi}^*(\xi, s, p) = \overline{\varphi}^*(s, p)$ and the matrix elements are calculated by double integration over the free cross-sectional area D of a channel

$$A_{jk} = -\int \left(\frac{\partial^2 \psi_k}{\partial \xi^2} + \beta \frac{\partial^2 \psi_k}{\partial \eta^2} \right) \psi_j \, d\sigma = \int \left(\frac{\partial \psi_k}{\partial \xi} \frac{\partial \psi_j}{\partial \xi} + \beta \frac{\partial \psi_k}{\partial \eta} \frac{\partial \psi_j}{\partial \eta} \right) d\sigma = A_{kj} > 0 \,, \quad B_{jk} = \int w \psi_k \, \psi_j d\sigma = B_{kj} > 0 \,,$$

$$C_{jk} = \int \psi_j \, \psi_k d\sigma = C_{kj} > 0 \,, \quad F_j = \int w \psi_j d\sigma \,, \quad N_j = \int \psi_j d\sigma \,, \quad E_j = \int \psi_0 \, \psi_j d\sigma \,, \quad d\sigma = d\xi d\eta \,. \tag{5}$$

The alternative of selecting the coordinate functions $\psi_k(\xi, \eta)$ is confirmed for fulfillment of only the homogeneous boundary conditions of the problem formulated, and for (2) we must have $[\psi_k]_\Gamma = 0$. This allows us to attain agreement with the boundary conditions in representation (3). Since the matrices $\|A\|$, $\|B\|$, and $\|C\|$ are symmetric and positive, the roots of the algebraic equations $|A + sB| = 0$, $|A + pC| = 0$, $-s_1^{(n)}$, $-s_2^{(n)}$, ..., $-s_n^{(n)}$, $-p_1^{(n)}$, $-p_2^{(n)}$, ..., $-p_n^{(n)}$, will be real and negative ($s_k^{(n)} > 0$, $p_k^{(n)} > 0$).

With steady-state heat loadings where $\varphi(X, \mathrm{Fo}) = \varphi(X)$, $f(X, \mathrm{Fo}) = f(X)$, and $\varphi_0(\mathrm{Fo}) = T_0$, in Eq. (1) we have $\partial T/\partial \mathrm{Fo} = 0$ and, according to the Cramer formula, from system (4) at $p = 0$ we obtain the following expression:

$$a_k^*(s) = \frac{\Delta_k^{(F)}(s)\,[T_0 - s\varphi^*(s)]}{\Delta(s)} + \frac{q_v h^2}{\lambda} \frac{\Delta_k^{(E)}(s) f^*(s)}{\Delta(s)} \,,$$

where $\Delta_k^{(M)}(s) = \sum_{j=1}^{n} M_j \Delta_{jk}(s)$, $\Delta_{jk}(s)$ are the algebraic complements of the determinant $\Delta(s) = |A + sB|$,

$$f^*(s) = \int_0^\infty f(X) \exp(-sX)\, dX \,; \quad \varphi^*(s) = \int_0^\infty \varphi(X) \exp(-sX)\, dX \,,$$

i.e., $T^*(\xi, \eta, s)$ is already the Laplace transform. By decomposing the proper fractions $\Delta_k^{(F)}(s)/\Delta(s)$ and $\Delta_k^{(E)}(s)/\Delta(s)$ into partial fractions in simple roots of the denominator, we obtain a synthesis of the elements of the matrix-response $\|a^*(s)\|$ to sums of blocks of elementary thermoinertial links in the following form:

$$a_k^*(s) = \sum_{i=1}^{n} \frac{\Delta_k^{(F)}(-s_i^{(n)})}{\Delta'(-s_i^{(n)})} \frac{T_0 - s\varphi^*(s)}{s + s_i^{(n)}} + \frac{q_v h^2}{\lambda} \sum_{i=1}^{n} \frac{\Delta^{(E)}(-s_i^{(n)})}{\Delta'(-s_i^{(n)})} \frac{f^*(s)}{s + s_i^{(n)}} \,, \quad \Delta' = \frac{d\Delta}{ds} \,. \tag{6}$$

We set $T_0 - s\overline{\varphi}^*(s) \doteqdot \Phi(X)$; then, with the aid of the convolution theory, representation (3) in the domain of inverse transforms can be written, after permutation of the orders of summation with respect to k and i, as

$$T_n(\xi, \eta, X) = \varphi(X) + \sum_{i=1}^{n} \int_0^X \Phi(\gamma) \exp[-s_i^{(n)}(X - \gamma)]\, d\gamma\, \psi_i^{(n)}(\xi, \eta) +$$

$$+ \frac{q_v h^2}{\lambda} \sum_{i=1}^{n} \int_0^X f(\gamma) \exp[-s_i^{(n)}(X - \gamma)]\, d\gamma\, \varphi_i^{(n)}(\xi, \eta) \,,$$

where

$$\psi_i^{(n)}(\xi, \eta) = \sum_{k=1}^{n} \frac{\Delta_k^{(F)}(-s_i^{(n)})}{\Delta'(-s_i^{(n)})} \psi_k(\xi, \eta) ; \quad \varphi_i^{(n)}(\xi, \eta) = \sum_{k=1}^{n} \frac{\Delta_k^{(E)}(-s_i^{(n)})}{\Delta'(-s_i^{(n)})} \psi_k(\xi, \eta) ,$$

As the wall temperature changes continuously with the temperature at the channel inlet ($\lim_{X \to 0} \varphi(X) = T_0$) it follows that $-\Phi(X) = \dfrac{d\varphi}{dX}$ and the integral under the sign of the first sums is transformed to the form

$$\int_0^X \Phi(\gamma) \exp[-s_i^{(n)}(X-\gamma)]\, d\gamma = [T_0 \exp(-s_i^{(n)}X) - \varphi(X)] + s_i^{(n)} \int_0^X \varphi(\gamma) \exp[-s_i^{(n)}(X-\gamma)]\, d\gamma .$$

The results of systematic use of the solving algorithm show that sufficient accuracy of calculation of the temperature fields is attained in the second or third approximations, while for heat-releasing fluids the method of the optimum selection of basis coordinate functions leads to a high accuracy of representation of the temperature already in the first approximation [1, 3, 4].

For the sake of simplicity and illustration, we will consider unsteady heat transfer in a round tube ($m = 1,\ 0 \le \xi = r/R = 1$) and a plane channel ($m = 0,\ -1 \le \xi = x/R \le 1$) as a solution of the problem

$$\frac{\partial T}{\partial \mathrm{Fo}} + w(\xi, m) \frac{\partial T}{\partial X} = \frac{1}{\xi^m} \frac{\partial}{\partial \xi}\left(\xi^m \frac{\partial T}{\partial \xi}\right) + \frac{q_v R^2}{\lambda} \psi_0(\xi) f(X, \mathrm{Fo}) , \quad [T(\xi, X, \mathrm{Fo})]_{\mathrm{Fo}=0} = T_0 ; \tag{7}$$

$$[T(\xi, X, \mathrm{Fo})]_{X=0} = \varphi_0(\mathrm{Fo}) , \quad \left[\frac{\partial T}{\partial \xi} + \mathrm{Bi}\, T(\xi, X, \mathrm{Fo})\right]_{\xi=1} = \mathrm{Bi}\,[\varphi(X, \mathrm{Fo}) + q(X, \mathrm{Fo})/\alpha] , \quad \left(\frac{\partial T}{\partial \xi}\right)_{\xi=0} = 0 . \tag{8}$$

The temperature in the Laplace–Carson transforms is found in the variety

$$\overline{T}_n^*(\xi, s, p) = \overline{\Phi}^*(s, p) + \sum_{k=1}^{n} \overline{a}_k^*(s, p)\, \psi_k(\xi) , \quad \psi_k(\xi) = \frac{\mathrm{Bi} + 2k}{\mathrm{Bi}} - \xi^{2k} , \quad \overline{\Phi}^* = \overline{\varphi}^* + \overline{q}^*/\alpha \tag{9}$$

and the coefficients in system (4) are calculated from the formulas

$$A_{jk} = -\int_0^1 \frac{\partial}{\partial \xi}\left(\xi^m \frac{\partial \psi_k}{\partial \xi}\right) \psi_j d\xi = \int_0^1 \frac{\partial \psi_k}{\partial \xi} \frac{\partial \psi_j}{\partial \xi} \xi^m d\xi + \mathrm{Bi}\, \psi_k(1)\, \psi_j(1) = A_{kj} > 0 ; \quad C_{jk} = \int_0^1 \psi_j \psi_k \xi^m d\xi = C_{kj} > 0 ;$$

$$B_{jk} = \int_0^1 w \psi_k \psi_j \xi^m d\xi = B_{kj} > 0 ; \quad F_j = \int_0^1 w \psi_j \xi^m d\xi ; \quad N_j = \int_0^1 \psi_j \xi^m d\xi , \quad E_j = \int_0^1 \psi_0 \psi_j \xi^m d\xi . \tag{10}$$

The equations of energy (heat) transfer (1) and (7) are written from a system of Navier–Stokes equations as particular cases of the motion of media in channels. They will become closed if a stabilized velocity field is found by solving the Poisson equations

$$\frac{\partial^2 W}{\partial \xi^2} + \beta \frac{\partial^2 W}{\partial \eta^2} = \frac{h^2}{\mu} \frac{\partial \mathcal{P}}{\partial z} , \quad \frac{1}{\xi^m} \frac{\partial}{\partial \xi}\left(\xi^m \frac{\partial W}{\partial \xi}\right) = \frac{R^2}{\mu} \frac{\partial \mathcal{P}}{\partial z} , \quad \beta = \frac{h^2}{b^2} , \quad m = 0; 1 , \tag{11}$$

with the zeroth boundary conditions over the wetted surface, which are also written from the Navier–Stokes system. We would like to recall that the velocity $w = W/w_0$ enters into (1) and (7).

In all the known solutions of the problems of heat transfer in straight channels, beginning with the classical investigations of Gretz and Nusselt, stabilized velocities have been considered only in isothermal flows, i.e., with the aid of the solution of Eqs. (11) for constant coefficients of dynamic viscosity $\mu\left(\frac{1}{\mu}\frac{\partial \mathcal{P}}{dz} = \text{const}\right)$. Meanwhile, the process of heat transfer occurs almost without exception with nonuniform temperature distributions in a fluid flow, i.e., under conditions of nonisothermal flow even in stabilized regimes. Substantiation of the solutions of Eq. (1) without account for the contribution of heat conduction along the channel, when the heat-technology system becomes inertial over the unilateral coordinate X, and the influence of this term on heat transfer in liquid-metal heat-transfer agents are discussed in [5].

Let us assume that in channels with a two-dimensional cross-sectional profile the steady-state temperature distribution, which depends on ξ or ξ, η, is established in a fluid flow with time and along the tube. Since the dynamic viscosity μ mainly depends on temperature, with such a temperature distribution the coefficients $k = \mu^{-1}$ of the motive force of the fluid on the right-hand sides of the Poisson equations will be a function of the coordinates ξ, η as well. Let the coefficient $k = \mu^{-1}$ be interpolated by the function $\varphi(\xi, \delta)$ in the form $k = (\xi, \delta) = \varphi(\xi, \delta)/\mu_0$, where $\varphi(0, \delta) = 1$, and the steady-state motion of a viscous fluid in a round tube be determined by a solution of the equation

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial W}{\partial\xi}\right) = -\frac{R^2\varphi(\xi, \delta)}{\mu_0}\frac{\Delta\mathcal{P}}{l}, \quad \frac{\partial\mathcal{P}}{\partial z} = -\frac{\Delta\mathcal{P}}{l}. \tag{12}$$

We set $\varphi(\xi, \delta) = 1 + \delta\xi$; then the linear change in the force of fluid transfer along the running radius ξ forms the velocity profile of the heating fluid ($T_w > T_0$) for $\delta > 0$ and of the cooling fluids ($T_w < T_0$) for $\delta < 0$. When $[W]_\Gamma = 0$ the equation has the solution

$$W(\xi, \delta) = \frac{R^2}{36\mu_0}\frac{\Delta\mathcal{P}}{l}\,[9 + 4\delta - (9\xi^2 + 4\delta\xi^3)] .$$

From the condition of conservation of the incompressible-fluid mass $\pi R^2 w_0 = \int_0^{2\pi} d\varphi \int_0^1 W\xi d\xi$, we find w_0; then the volumetric flow rate of the fluid and the velocity of the nonisothermal flow are equal:

$$V = \pi R^2 w_0 = \frac{\pi R^4 (45 + 24\delta)}{360\mu_0}\frac{\Delta\mathcal{P}}{l}, \quad w(\xi, \delta) = \frac{W}{w_0} = \frac{2(1 + 0.444\delta)}{1 + 0.533\delta}\left(1 - \frac{9\xi^2 + 4\delta\xi^3}{9 + 4\delta}\right). \tag{13}$$

In heating of the fluid ($\delta > 0$, $T_w > T_0$), the maximum velocities in the flow core are as follows:

$$W(0, 0.2) = 1.968w_0; \quad W(0, 0.6) = 1.919w_0; \quad W(0, 0.8) = 1.900w_0; \quad W(0, 0.98) = 1.884w_0$$

and in cooling ($\delta < 0$, $T_w < T_0$) they are

$$W(0, -0.2) = 2.040w_0; \quad W(0, -0.6) = 2.156w_0; \quad W(0, -0.8) = 2.247w_0; \quad W(0, -0.98) = 2.364w_0 .$$

In cooling of the fluid via the surface of the round tube, an inflection point exists in the stabilized velocity profile [6], and at this point $d^2W/d\xi^2 = 0$. From the solution (13) we have $d^2W/d\xi^2 = 18 + 24\delta\xi = 0$, whence, as it must, the inflection point is found just for $\delta < 0$. For $\delta = -0.8$ inflection occurs at $\xi = 0.95$,

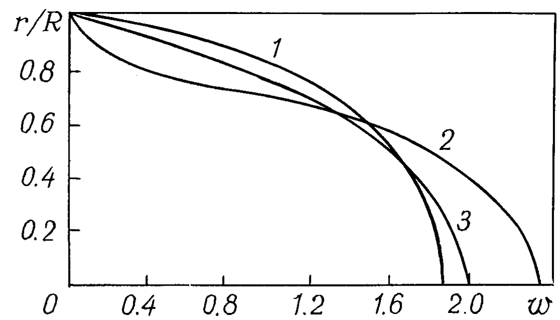

Fig. 1. Velocity profiles of nonisothermal flow in a round tube: 1) heating of the fluid for μ_w/μ_0 = 0.5; 2) cooling of the fluid for μ_w/μ_0 = 50; 3) isothermal flow for μ_w/μ_0 = 1.

while for $\delta = -0.98$ when $\mu_w/\mu_0 = \mu(1, \delta)/\mu(0, \delta) = 50$ inflection in the velocity profile occurs in the zone $\xi = 0.77$. The velocity profiles for $\delta = 0.98$, $\delta = 0$, and $\delta = -0.98$, which correspond to $\mu_w/\mu_0 = \mu(1)/\mu(0) = 0.5$, $\mu_w/\mu_0 = 1$, and $\mu_w/\mu_0 = 50$, are shown in Fig. 1.

With another form of change in the local distribution of the dynamic viscosity $\mu(\xi, \delta) = \mu_0 \exp(-\delta\xi)$ where, as in the case $\delta > 0$, the fluid is heated and for $\delta < 0$ is cooled, integration of Eq. (12) with the right-hand side $-R^2\mu_0^{-1} \exp(\delta\xi)\Delta\mathcal{P}/l$ yields

$$W(\xi, \delta) = \frac{w_0}{2H(\delta)}\left[\frac{\exp(\delta) - \exp(-\delta\xi)}{\delta^2} - \frac{1-\xi}{\delta} - \frac{1-\xi^2}{4} - \delta f(\xi, \delta)\right]; \quad f = \frac{1-\xi^3}{3\cdot 3\,!} + \frac{1-\xi^4}{4\cdot 4\,!} + \ldots ; \tag{14}$$

$$w_0 = \frac{2R^2}{\mu_0}\frac{\Delta\mathcal{P}}{l}H(\delta); \quad H(\delta) = \frac{\exp(\delta)(3\delta^2 - 6\delta + 6) - (6 + \delta^3)}{6\delta^4} - \frac{1}{16} - \delta\int_0^1 f(\xi, \delta)\,\xi d\xi ,$$

from which and also from (13) we obtain the known Poiseuille formulas for isothermal flow

$$\lim_{\delta\to 0} W(\xi, \delta) = W(\xi, 0) = 2w_0(1 - \xi^2), \quad w_0 = \frac{R^2}{8\mu_0}\frac{\Delta\mathcal{P}}{l} .$$

Let us consider solutions of the first equation of (11) with variable viscosities μ, for which the enforced flow of the fluid with constant pressure gradient $\partial\mathcal{P}/\partial z = -\Delta\mathcal{P}/l$ = const is initiated by the variable force

$$\frac{h^2}{\mu}\frac{\partial\mathcal{P}}{\partial z} = -\frac{h^2}{\mu_0}\frac{\Delta\mathcal{P}}{l}\varphi(\xi, \eta, \delta) .$$

In the case of laminar isothermal or even nonisothermal flows of the fluids in channels with two-dimensional profiles of the cross sections, the velocity fields will depend in many respects on the geometry of the free cross-sectional area. Therefore, in the variety of the representation of the solutions in alternative spaces

$$W_n(\xi, \eta) = \sum_{k=1}^{n} a_k\,\psi_k(\xi, \eta), \quad [\psi_k(\xi, \eta)]_\Gamma = 0 , \tag{15}$$

it is more expedient and natural to use the basis coordinate functions $\psi_k(\xi, \eta) = \omega(\xi, \eta)\xi^{(k-1)}\eta^{2(k-1)}$, where $\omega(\xi, \eta)$ is the composite boundary function for the channels mentioned at the beginning of the paper. In structure of the representation, $\omega(\xi, \eta) > 0$ inside the channel and is equal to zero over the wetted surface. We determine the solutions for such variable forces of fluid motion, under the action of which the exact velocity fields are expressed by one spectrum along the first principal axis in the variety of representation (15). For a channel of isosceles triangular cross section with height h and base $2b$: $D\left\{y \le \frac{b}{h} x, y \ge -\frac{b}{h} x, 0 \le x \le h\right\}$ the composite boundary function is

$$\omega(\xi, \eta) = (\xi^2 - \eta^2)(1 - \xi), \quad 0 \le \xi = \frac{x}{h} \le 1; \quad -1 \le \eta = \frac{y}{b} \le 1 .$$

If, in the first equation of (11), we set

$$\frac{h^2}{\mu} \frac{\partial \mathcal{P}}{\partial z} = -\frac{h^2 [(\delta - 1) + \xi (3 - \delta)]}{2\mu_0} \frac{\Delta \mathcal{P}}{l}, \quad \delta = \beta, \tag{16}$$

then the application of the orthogonal projection of the discrepancy to the solution $W(\xi, \eta) = a_1 \omega(\xi, \eta)$ yields the equality

$$\int_0^1 \int_0^{\xi} \left[\left(\frac{\partial^2 \omega}{\partial \xi^2} + \beta \frac{\partial^2 \omega}{\partial \eta^2}\right) + \frac{h^2 [(\delta - 1) + \xi (3 - \delta)]}{2\mu_0} \frac{\Delta \mathcal{P}}{l}\right] \omega(\xi, \eta)\, d\xi d\eta = 0 ,$$

from which we find $a_1 = \frac{h^2}{4\mu_0} \frac{\Delta \mathcal{P}}{l}$ and, expressing a_1 in terms of the mean velocity w_0, we arrive at

$$W(\xi, \eta) = 15 w_0 (\xi^2 - \eta^2)(1 - \xi), \quad w_0 S_D = \frac{bh^3}{60\mu_0} \frac{\Delta \mathcal{P}}{l}, \quad \max W = W\left(\frac{2}{3}, 0\right) = 2.22 w_0 \tag{17}$$

which is an equally exact solution for all the channels with isosceles triangular cross sections where the nonuniform force of fluid motion is prescribed in the form of (16). For an equilateral triangular cross section $\delta = \beta = h^2/b^2 = 3$ and a mass-moving force, (16) passes to the constant value $-(\mu_0^{-1} h^2 \Delta \mathcal{P}/l = \text{const})$, i.e., in all triangular channels just with three equal walls the stabilized velocity field of the isothermal flow is exactly expressed in terms of the composite boundary function. Inside the channel with the profile of a right triangle ($\beta = 1$), the exact value of the velocity (17) is created by the force $-\mu_0^{-1} h^2 \xi \Delta \mathcal{P}/l$.

For the rectangular channel $D\{-h \le x \le h, -b \le y \le b\}$ with the composite boundary function $\omega(\xi, \eta) = (1 - \xi^2)(1 - \eta^2)$, the exact solution of the first equation of (11) with the force of fluid transfer $-h^2 \mu_0^{-1} [(1 - \eta^2) + \beta(1 - \xi^2)] \Delta \mathcal{P}/l$ is

$$W(\xi, \eta) = 2.25 w_0 (1 - \xi^2)(1 - \eta^2), \quad w_0 = \frac{h^2}{4.5\mu_0} \frac{\Delta \mathcal{P}}{l}, \quad \max W = 2.25 w_0 .$$

Curiously, the dimensionless velocity $w(\xi, \eta) = W/w_0$ is a result of the superposition (multiplication) of two profiles of Poiseuille velocities $w(\xi) = 1.5(1 - \xi^2)$ and $w(\eta) = 1.5(1 - \eta^2)$ in plane-parallel channels.

Inside a cylindrical tube with an elliptic profile of the free cross-sectional area, the velocity field of the stabilized isothermal flow is exactly expressed by the composite boundary function $\omega(\xi, \eta) = 1 - \xi^2 - \eta^2$ in the form

$$W(\xi, \eta) = 2 w_0 (1 - \xi^2 - \eta^2), \quad w_0 = \frac{h^2}{4\mu_0 (1 + \beta)} \frac{\Delta \mathcal{P}}{l} .$$

In all the solutions found, the lines of equal velocities v_i = const (isotachs) related to the mean velocity w_0 are determined by the equations $W/w_0 = w(\xi, \eta) = v_i$ = const, $i = 1, n$, where $0 \le v_i \le 2.2$ for an isosceles triangular cross section, $0 \le v_i \le 2.25$ for a rectangular cross section, and $0 \le v_i \le 2$ for an elliptic tube. The isotachs in a geometric set are quasisimilar to the closed composite boundary line, i.e., similar to the profile of the free cross-sectional area of the channel. The zero isotach on the wetted surface is equal to $\omega(\xi, \eta) = 0$.

We proceed to an investigation of the processes of heat transfer with different forms of stabilized velocities under the conditions of isothermal flow with allowance for heating or cooling of the fluid via the channel walls. We introduce the value of the velocity (13) into formulas (10) and calculate the coefficients of the determining system for a round tube with boundary conditions of the first kind, i.e., with $\psi_k(\xi) = 1 - \xi^{2k}$ and $m = 1$; then

$$B_{jk} = \frac{2\,(1 + 0.444\delta)}{1 + 0.533\delta}\left[\frac{kj\,(k + j + 2)}{2\,(k + 1)\,(j + 1)\,(k + j + 1)} - \frac{1}{9 + 4\delta}\left(\frac{9kj\,(k + j + 4)}{4\,(k + 2)\,(j + 2)\,(k + j + 2)} + \right.\right.$$

$$\left.\left. + \frac{32\delta kj\,(k + j + 5)}{5\,(2k + 5)\,(2j + 5)\,(2k + 2j + 5)}\right)\right] = B_{kj} > 0\,;$$

$$A_{kj} = \frac{2kj}{k + j} = A_{jk} > 0\,; \quad C_{jk} = \frac{kj\,(k + j + 2)}{2\,(k + 1)\,(j + 1)\,(k + j + 1)} = C_{kj} > 0\,;$$

$$F_j = \frac{2\,(1 + 0.444\delta)}{1 + 0.533\delta}\left[\frac{j}{2\,(j + 1)} - \frac{1}{9 + 4\delta}\left(\frac{9j}{4\,(j + 2)} + \frac{8\delta j}{5\,(2j + 5)}\right)\right];$$

$$N_j = \frac{j}{2\,(j + 1)}\,; \quad E_j = N_j\,; \quad \psi_0\,(\xi) = 1\,.$$

Representation of the matrix elements in terms of the subscripts allows us to write the determining system (4) of any order in explicit form for the fixed parameter δ. Using the formulas of synthesis of the elements of the matrix-response $\|a^*(s)\|$ in the form (6), we determine $a_k(X)$ with concrete steady-state heat loadings along the tube on the wall $\varphi(X)$ and the functions of loading of the internal sources $f(X)$, and the solution is found in the form

$$T\,(\xi, X) = \varphi\,(X) + \sum_{k=1}^{n} a_k\,(X)\,(1 - \xi^{2k})\,.$$

From the truncated system of first order without the internal source ($\psi_0(\xi) = 0$), we have

$$a_1^*\,(s) = \frac{[T_0 - s\varphi^*\,(s)]\,F\,(\delta)}{s + s_1^{(1)}\,(\delta)}\,, \quad s_1^{(1)}\,(\delta) = \frac{4\,(1 + 0.533\delta)}{1 + 0.444\delta}\,, \quad F\,(\delta) = \frac{4\,(1 + 0.5143\delta)}{3\,(1 + 0.5342\delta)}\,,$$

from which at a constant wall temperature ($\varphi(X) = T_{\rm w}$), when the fluid is heated ($T_{\rm w} > T_0$, $\delta > 0$) or cooled ($T_{\rm w} < T_0$, $\delta < 0$), the temperature changes, including heat transfer in the case of Poiseuille isothermal flow of the fluid ($\delta = 0$), are given by the single formula

$$T\,(\xi, X, \delta) = T_{\rm w} + F\,(\delta)\,(T_0 - T_{\rm w})\,(1 - \xi^2)\,\exp\,[-\,s_1^{(1)}\,(\delta)\,X]\,. \tag{18}$$

For δ = 0.2, 0.9, δ = 0, and δ = –0.2, –0.9 the relative excess temperature fields are reduced to the expressions

$$\Theta(\xi, X, 0.2) = \frac{T - T_w}{T_0 - T_w} = 1.336\,(1 - \xi^2)\exp(-4.023X)\,;\quad \Theta(\xi, X, 0.9) = 1.342\,(1 - \xi^2)\exp(-4.229X)\,;$$

$$\Theta(\xi, X, 0) = \frac{T - T_w}{T_0 - T_w} = 1.333\,(1 - \xi^2)\exp(-4.000X)\,; \tag{19}$$

$$\Theta(\xi, X, -0.2) = \frac{T - T_w}{T_0 - T_w} = 1.331\,(1 - \xi^2)\exp(-3.974X)\,;\quad \Theta(\xi, X, -0.9) = 1.314\,(1 - \xi^2)\exp(-3.469X)\,.$$

Solutions in the second and subsequent approximations should be refined for the concrete parameter δ. For the heated fluid with δ = 0.2 in the second approximation we have

$$\Theta(\xi, X, 0.2) = (1.448 - 2.229\xi^2 + 0.781\xi^4)\exp(-3.687X) + (-0.623 + 3.889\xi^2 - 3.266\xi^4)\exp(-35.880X)\,.$$

The values of the temperature to the fourth order of approximations of isothermal (δ = 0) flow are found in [1], from which it follows in the second approximation that

$$\Theta(\xi, X, 0) = (1.437 - 2.228\xi^2 + 0.791\xi^4)\exp(-3.670X) + (-0.604 + 3.896\xi^2 - 3.292\xi^4)\exp(-36.330X)\,.$$

Since the theoretical studies of the rate of convergence of the approximate solutions found by the suggested method allow us to assert that a sufficient accuracy of calculation is attained already from the second approximation, based on the results obtained we can draw conclusions on the regularities of heat transfer in the cases where the stabilized velocity is formed by a nonisothermal flow. From the solutions it follows that in heating ($T_w > T_0$, $\delta > 0$) the rate of exponential stabilizations of the temperature fields, the local heat fluxes, and the Nusselt numbers are higher, while in cooling ($T_w < T_0$, $\delta < 0$) they are lower than in the case of Poiseuille isothermal flow of the fluid. Although for the velocity profile (13) such a regularity is manifested still weakly, for more abrupt changes in the moving force, for instance, in the case of fluid motion with the velocity (14) for $\delta = \pm 3$, this difference becomes already more pronounced.

We determine the change in the temperature in the fluid flow with stabilized velocity (13) with a linear rise in the temperature over the tube surface in the form $\varphi(X) = T_0 + \Delta\tilde{T}X$. Behind the portion of thermal stabilization the temperature field becomes self-similar and is expressed in the form

$$T(\xi, X, \delta) = T_0 + \Delta\tilde{T}X + \Phi(\xi, \delta)\,,\quad \Delta\tilde{T} = \mathrm{Pe}\,R\Delta T\,,\quad \Delta T = \frac{d\varphi}{dz}\,,$$

where the function $\Phi(\xi, \delta)$ is found by solving the problem

$$w(\xi, \delta)\,\Delta\tilde{T} = \frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\Phi}{d\xi}\right),\quad [\Phi(\xi, \delta)]_{\xi=1} = 0\,,\quad \left(\frac{d\Phi}{d\xi}\right)_{\xi=0} = 0$$

and is equal to

$$\frac{18\,(1 + 0.533\delta)}{\Delta\tilde{T}}\,\Phi(\xi, \delta) = \psi(\xi, \delta) = (6.75 + 3.36\delta) - (9 + 4\delta)\,\xi^2 + 2.25\xi^4 + 0.64\delta\xi^5\,.$$

According to the method of selection of the optimum basis coordinate functions [1], we should take $\psi_1(\xi) = \psi(\xi, \delta)$; then from the equation

$$a_1^*(s)\,(A_{11} + B_{11}s) = [T_0 - s\varphi^*(s)]\,F_1\,,\quad \varphi^*(s) = T_0/s + \Delta\tilde{T}/s^2\,,$$

we find

$$a_1^*(s) = -\frac{F_1\Delta\tilde{T}}{s\,(s + s_1^{(1)})\,B_{11}} = -\frac{F_1\Delta\tilde{T}}{A_{11}}\left(\frac{1}{s} - \frac{1}{s + s_1^{(1)}(\delta)}\right),\quad s_1^{(1)}(\delta) = \frac{A_{11}}{B_{11}}\,,$$

and the solution is reduced to the form

$$T(\xi, X, \delta) = T_0 + \Delta\tilde{T}X - \frac{\Delta\tilde{T}}{18\,(1 + 0.533\delta)}\,[6.75 + 3.36\delta - (9 + 4\delta)\,\xi^2 + 2.25\xi^4 +$$

$$+ 0.64\delta\xi^5]\,[1 - \exp(-s_1^{(1)}(\delta)\,X)]\,, \tag{20}$$

where, with the purpose of decreasing the number of intermediate calculations, the coefficients A_{11}, B_{11}, and F_1 should be calculated from formulas (10) for $m = 1$ with the concrete parameter δ. On the whole, after the interval of the transient regime this expression will coincide with the exact solution with the structure of temperature representation. For a laminar isothermal flow with the velocity $w(\xi, 0) = 2(1 - \xi^2)$ we have

$$T(\xi, X, 0) = T_0 + \Delta\tilde{T}X - \frac{\Delta\tilde{T}}{8}\,(3 - 4\xi^2 + \xi^4)\,[1 - \exp(-3.729X)]\,, \tag{21}$$

while with the velocity $w(\xi, 0.8) = 1.9(1 - 0.738\xi^2 - 0.262\xi^3)$ the temperature field is represented by the formula

$$T(\xi, X, 0.8) = T_0 + \Delta\tilde{T}X - 0.475\Delta\tilde{T}\,(0.774 - \xi^2 + 0.184\xi^4 + 0.042\xi^5)\,[1 - \exp(-4.444X)]\,. \tag{22}$$

Behind the tube inlet from the solution (21) we obtain the exact self-similar temperature representation which was found by Eagle and Fergusson [7].

Performing differentiation of the solution (21), we determine the heat flux in the case of isothermal flow

$$q(X) = -\frac{\lambda}{R}\left(\frac{\partial T}{\partial \xi}\right)_{\xi=1} = \frac{\lambda\Delta\tilde{T}}{2R}\,[1 - \exp(-3.729X)]\,,\quad \tilde{q}(X) = \frac{q(X)\,R}{\lambda\,\Delta\tilde{T}} = 0.5\,[1 - \exp(-3.729X)]\,,$$

and from the temperature field in the second approximation we obtain

$$\tilde{q}(X) = 0.5\,[1 - 0.734\exp(-3.662X) - 0.266\exp(-30.863X)]\,.$$

From the solution (22) with the fluid heated by the linear rise in the wall temperature with allowance for the nonisothermicity of the flow, when the coefficient $k = \mu^{-1}$ at the transfer force is interpolated by the linear function of the running radius $\xi = r/R$, the specific heat flux is

$$\tilde{q}(X) = 0.5\,[1 - \exp(-4.444X)]\,.$$

The rate of exponential stabilization of the temperature in exact solutions in the case of isothermal flow is determined by the first eigenvalue $s_1(0) = 3.658$. In the approximate solution (21), this rate is $s_1^{(1)}(0)$

= 3.729 and it is closer to the exact value than in (19). This is because the temperature (21) was found in the space with the optimum coordinate function $\psi_1(\xi) = 3 - 4\xi^2 + \xi^4$. The heat-transfer rate $s_1^{(1)}(0.8)$ determines the temperature in the fluid flow with the more filled velocity profile $w(\xi, 0.8) = 1.9(1 - 0.738\xi^2 - 0.262\xi^3)$ than in the Poiseuille flow, i.e., we must have $s_1(0.8) > 3.658$. Therefore, we may consider that the value $s_1^{(1)}(0.8) = 4.444$, just as the value $s_1^{(1)}(0) = 3.729$, is found with sufficient accuracy.

Similar studies of heat transfer can be carried out for problems with external loadings, at which the wall temperature $\varphi(X)$ on a small initial portion of the channel becomes the linear function $T_0 + \Delta\tilde{T}X$. The limiting exact self-similar solutions will depend on the kinds of velocities of nonisothermal flows created by different transfer forces $\frac{h^2}{\mu}\frac{\partial \mathcal{P}}{\partial z} = \frac{h^2\varphi(\xi,\delta)}{\mu_0}\frac{\partial \mathcal{P}}{\partial z}$.

Now we will provide calculation of heat transfer in the flow of a heat-releasing fluid with allowance for nonisothermal flow when the coefficient $k = \mu^{-1}$ of the moving force of transfer is interpolated by a quadratic function, i.e.,

$$\frac{1}{\mu}\frac{\partial \mathcal{P}}{\partial z} = -\frac{(1 + \delta\xi^2)}{\mu_0}\frac{\Delta \mathcal{P}}{l} .$$

The solution of the second equation of (11) with such a right-hand side for a round tube is

$$W(\xi, \delta) = \frac{3w_0}{2(3+\delta)}\left[4 + \delta - (4\xi^2 + \delta\xi^4)\right] , \quad w_0 = \frac{R^2(3+\delta)}{24\mu_0}\frac{\Delta \mathcal{P}}{l} .$$

From the equation $d^2W/d\xi^2 = 2 + 3\delta\xi^2 = 0$ it follows that in the velocity profile the inflection point exists for $\delta < 0$, i.e., only the cooling fluid has such a point. For $\delta = -0.8$ the inflection point is in the zone $\xi = 0.9$, while for $\delta = -0.9$ it is in the zone $\xi = 0.85$. The dimensionless velocities related to the mean value of w_0 for the three δ are

$$w(\xi, 0) = 2(1 - \xi^2) ; \quad w(\xi, -0.8) = 2.182(1 - 1.25\xi^2 + 0.25\xi^4) ; \quad w(\xi, -0.9) = 2.214(1 - 1.29\xi^2 + 0.29\xi^4) .$$

Since for $|\delta| < 1$ the following expansion in a power series holds true:

$$\mu = \frac{1}{k} = \frac{\mu_0}{1 + \delta\xi^2} = \mu_0(1 - \delta\xi^2 + \delta^2\xi^4 - \delta^3\xi^6 + ...) ,$$

the coefficient of dynamic viscosity can also be represented by the quadratic function $\mu(\xi, \delta) = \mu_0(1 - \delta\xi^2)$, where $\delta < 0$ in cooling of the fluid and $\delta > 0$ in its heating.

The temperature fields in the optimum Riemann space for a round tube with the internal source $q_v(\xi) = q_v$ = const in removal of heat to the external medium with a constant temperature equal to the fluid temperature at the tube inlet T_0 for the cases $f(X) = 1$ and $f(X) = 1 - \exp(-\mathrm{Pd}\,X)$ are reduced to the expressions

$$T(\xi, X, \delta) = T_0 + \frac{q_vR^2}{4\lambda}\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2\right)\left[1 - \exp(-s_1^{(1)}(\mathrm{Bi}, \delta)X)\right] ,$$

$$T(\xi, X, \mathrm{Bi}, \delta, \mathrm{Pd}) = T_0 + \frac{q_vR^2}{4\lambda}\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2\right)\left[1 - \frac{\mathrm{Pd}\exp(-s_1^{(1)}(\mathrm{Bi}, \delta)X) - s_1^{(1)}(\mathrm{Bi}, \delta)\exp(-\mathrm{Pd}\,X)}{\mathrm{Pd} - s_1^{(1)}(\mathrm{Bi}, \delta)}\right], \quad (23)$$

where

$$s_1^{(1)}(\text{Bi}, 0) = \frac{12\ \text{Bi}\ (\text{Bi} + 4)}{3\ \text{Bi}^2 + 16\ \text{Bi} + 24}\ ; \quad s_1^{(1)}(\text{Bi}, -0.9) = \frac{3.836\ \text{Bi}\ (\text{Bi} + 4)}{\text{Bi}^2 + 5.251\ \text{Bi} + 7.572}\ .$$

After entering the tube for $X \geq L$, where $\exp\,[-s_1^{(1)}(\text{Bi},\ \delta)X] \approx 0$ and $\exp\,(-PdX) \approx 0$, the following exact temperature distribution is established in the heat-releasing fluid:

$$T\,(\xi, \text{Bi}) = T_0 + \frac{q_v R^2}{4\lambda}\left(\frac{\text{Bi} + 2}{\text{Bi}} - \xi^2\right). \tag{24}$$

If, with the external linear temperature loadings on the wall, the self-similar exact representation of the temperature in the fluid flow depends on the velocity profiles of nonisothermal and even isothermal flows, the limiting solution (24) is no longer related to the character of the established velocity $w(\xi,\ \delta)$. However, the reduced length L will depend on the extent of filling of the velocity profile. As a more uniform filling is approached, this length decreases and for $w(\xi,\ \delta) = 1$ takes on its minimum value. As the velocity in the flow core increases, the length of the thermal-stabilization portion increases. For instance, for Bi = 0.5, 1, 4, 10, and ∞ the rate of stabilization of heat transfer in the isothermal flow $s_1^{(1)}(\text{Bi},\ 0)$ takes on the values 0.824, 1.395, 2.824, 3.471, and 4.000, while in the case of cooling we have 0.818, 1.378, 2.748, 3.359, and 3.836 for $s_1^{(1)}(\text{Bi},\ -0.9)$. Consequently, the length of the thermal-stabilization portion under cooling conditions is larger than for the same values of Bi in the problem with isothermal flow. From this viewpoint, it is of interest to solve a mathematical model of the process, in which the steady-state part of the distribution of the heat sources $q_v(\xi) = q_v(1 + \beta\xi^2)$ is prescribed for $\beta > 1$. Since for $\beta >> 1$ the fluid will be heated even in the case of heat removal owing to the substantial heat release in the near-wall layer, the moving force of heat transfer and the velocity $w(\xi,\ \delta)$ must be determined for $\delta > 0$. In the case of the parabolic distribution $q_v(\xi)$ the optimum coordinate function is $\psi_1(\xi) = [4(2 + \beta) + \text{Bi}(4 + \beta)/\text{Bi}] - (4\xi^2 + \beta\xi^4)$ and the temperature with boundary conditions of the first kind (Bi = ∞, $T_\text{w} = T_0$) for $\beta = \delta = 2$ is reduced to the form

$$T\,(\xi, X) = T_0 + \frac{q_v R^2}{8\lambda}\,(3 - 2\xi^2 - \xi^4)\,[1 - \exp\,(-\,4.308X)]\ ,$$

while for isothermal flow when $\delta = 0$ and $\beta = 2$ we have

$$T\,(\xi, X) = T_0 + \frac{q_v R^2}{8\lambda}\,(3 - 2\xi^2 - \xi^4)\,[1 - \exp\,(-\,4.416X)]\ .$$

For $\beta = 2$, a large amount of heat is released in the near-wall fluid layer, and it is rapidly removed, almost without conduction, into the external medium, which in the representations of the temperatures in terms of one principal thermoinertial link in the optimum Riemann spaces leads to the more overstated coefficients 4.308 and 4.416 than for $q_v(\xi) = q_v$ = const (4.000). We provide one more informative case where the parabolic distribution of the heat sources for $\beta = -1$ coincides with an accuracy of up to a constant factor with the velocity profile of isothermal flow. Then with a linear rise in the tube-wall temperature the procedure of implementation of the method in the variety of the representation $T^*(\xi, s) = \varphi^*(s) + \Sigma a_k^*(s)\psi_k(\xi)$ along the principal axis $\psi_1(\xi) = 3 - 4\xi^2 + \xi^4$ leads to the solution

$$T\,(\xi, X) = T_0 + \Delta\tilde{T}X + \left(\frac{q_v R^2}{16\lambda} - \frac{\Delta\tilde{T}}{8}\right)(3 - 4\xi^2 + \xi^4)\,[1 - \exp\,(-\,3.729X)]\ ,$$

which exactly satisfies the initial condition at the tube inlet and the temperature regime on the tube wall, while the self-similar solution coincides with the exact temperature. This property is retained for a series of problems in which the unsteady amplitude $f(\mathrm{Fo})$ of disturbance in the source $q_v(1-\xi^2)f(\mathrm{Fo})$ on the small time interval Fo changes to the steady one ($f(\mathrm{Fo}) = 1\ \forall\ \mathrm{Fo} \geq \mathrm{Fo}_1$).

Thus, the thermoinertial system of continuum mechanics which models the process of convective heat transfer in channels will have the highest rate of stabilization when fluid flow is of the piston type. Below, we provide the suggested method for such cases using the solution of the boundary-value problem (7) and (8) as an example. We set $w(\xi, \delta) = 1$, $\psi_k(\xi) = \frac{\mathrm{Bi}+2k}{\mathrm{Bi}} - \xi^{2k}$, and $\psi_0(\xi) = 1$; then the coefficients (10) are easily expressed in terms of the subscripts k and j, the Bi number, and the parameter m, which allows us to write the determining system (4) of any order in explicit form with a particular Bi number for a plane channel and a round tube individually. Since $\|B\| = \|C\|$ and $\|N\| = \|F\| = \|E\|$, the coefficients $\bar{a}_k^*(s, p)$ in the temperature representation (9) are equal to

$$\bar{a}_k^*(s, p) = \frac{\Delta_k^{(F)}(z)}{\Delta(z)}\left\{p\,[T_0 - \bar{\Phi}^*(s, p)] + s\,[\bar{\varphi}_0(p) - \bar{\Phi}^*(s, p)] + \frac{q_v R^2}{\lambda}\bar{f}^*(s, p)\right\},$$

where $\Delta(z) = |A + zC|$; $z = p + s$; $\Delta_k^{(F)}(z) = \sum_{j=1}^{n} F_j \Delta_{jk}(z)$; Δ_{jk} are the algebraic complements of the main determinant $\Delta(z)$. Let $-z_1^{(n)} < 0$, $-z_2^{(n)} < 0$, ..., $-z^{(n)} < 0$ be the roots of the equation $\Delta(z) = |A + zC| = 0$; then the positive numbers $z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)}$ form, obviously, a sequence of approximate eigenvalues for the problems of heat conduction in a plate or in a solid round cylinder. Expanding the proper fraction $\Delta_k^{(F)}(z)/\Delta(z)$ in simple poles, we obtain a synthesis of the elements of the matrix-response $\|\bar{a}^*(s, p)\|$ in the form

$$\bar{a}_k^*(s, p) = \sum_{i=1}^{n} \frac{\Delta_k(F)(-z_i^{(n)})}{\Delta'(-z_i^{(n)})}\left\{\frac{p\,[T_0 - \bar{\Phi}^*(s, p)]}{p + s + p_i^{(n)}} + \frac{s\,[\bar{\varphi}_0(p) - \bar{\Phi}^*(s, p)]}{p + s + p_i^{(n)}} + \frac{q_v R^2}{\lambda}\frac{\bar{f}^*(s, p)}{p + s + p_i^{(n)}}\right\}, \quad \Delta' = \frac{d\Delta}{dz},$$

where $p_i^{(n)} = z_i^{(n)}$. The projection $\bar{a}_k^*(s, p)$ in the temperature representation (9) along the ψ_k-axis of the alternative space is decomposed into sums of blocks of responses of elementary two-parameter thermoinertial links to the external $\bar{\varphi}_0(p)$, $\bar{\Phi}^*(s, p)$ and internal $\bar{f}^*(s, p)$ heat loadings. From the truncated system of first order we write

$$\bar{a}_1^*(s, p) = \frac{F_1}{C_{11}}\left\{\frac{p\,[T_0 - \bar{\Phi}^*(s, p)]}{p + s + p_1^{(1)}(\mathrm{Bi}, m)} + \frac{s\,[\bar{\varphi}_0(p) - \bar{\Phi}^*(s, p)]}{p + s + p_1^{(1)}(\mathrm{Bi}, m)} + \frac{q_v R^2}{\lambda}\frac{\bar{f}^*(s, p)}{p + s + p_1^{(1)}(\mathrm{Bi}, m)}\right\}, \tag{25}$$

where

$$p_1^{(1)}(\mathrm{Bi}, m) = \frac{A_{11}}{B_{11}} = \frac{\mathrm{Bi}\,(m+1)\,(m+5)\,[\mathrm{Bi} + (m+3)]}{2\,\mathrm{Bi}^2 + 2\,(m+5)\,\mathrm{Bi} + m^2 + 8m + 15}, \quad \frac{F_1}{C_{11}} = \frac{p_1^{(1)}(\mathrm{Bi}, m)}{2\,(m+1)}.$$

With spasmodic constant loadings ($\varphi_0(\mathrm{Fo}) = T_0$, $\Phi(X, \mathrm{Fo}) = T_{\mathrm{w}} + q/\alpha$, $f(X, \mathrm{Fo}) = 1$) we introduce the values of $\bar{\varphi}_0(p) = T_0$, $\bar{\Phi}^*(s, p) = T_{\mathrm{w}} + q/\alpha$, $\bar{f}^*(s, p) = 1$ into formula (25); then

$$\bar{a}_1^* (s, p) = \frac{p_1^{(1)} (\mathrm{Bi}, m)}{2 (m+1)} = \left\{ \frac{[T_0 - (T_\mathrm{w} + q/\alpha)] (p+s)}{p + s + p_1^{(1)} (\mathrm{Bi}, m)} + \frac{q_v R^2}{\lambda} \frac{1}{p + s + p_1^{(1)} (\mathrm{Bi}, m)} \right\} .$$

According to the inversion formula [2] for the double integral transformation

$$\frac{ap + bs}{A + Bs + Cp} \ddot{=} \begin{cases} a/C \exp \left(-\frac{A}{C} \mathrm{Fo} \right) & \text{for } X > \frac{B}{C} \mathrm{Fo} \\ b/B \exp \left(-\frac{A}{B} X \right) & \text{for } X < \frac{B}{C} \mathrm{Fo} \end{cases} , \quad \frac{1}{A + Bs + Cp} = \frac{1}{A} \left(1 - \frac{Bs + pC}{A + Bs + Cp} \right)$$

the combined relative excess temperature from $\bar{T}^*(\xi, s, p) = T_\mathrm{w} + q/\alpha + \bar{a}_1^* (s, p) \left(\frac{\mathrm{Bi} + 2}{\mathrm{Bi}} - \xi^2 \right)$ is reduced to the formula

$$\Theta (\xi, X, \mathrm{Fo}, \mathrm{Bi}, m) = \frac{T - (T_\mathrm{w} + q/\alpha)}{T_0 - (T_\mathrm{w} + q/\alpha)} = \frac{p_1^{(1)} (\mathrm{Bi}, m)}{2 (m+1)} \left(\frac{\mathrm{Bi} + 2}{\mathrm{Bi}} - \xi^2 \right) \begin{cases} \exp [- p_1^{(1)} (\mathrm{Bi}, m) \mathrm{Fo}] & \text{for } X > \mathrm{Fo} \\ \exp [- p_1^{(1)} (\mathrm{Bi}, m) X] & \text{for } X < \mathrm{Fo} \end{cases} +$$

$$+ \frac{q_v R^2}{2\lambda (m+1) [T_0 - (T_\mathrm{w} + q/\alpha)]} \left(\frac{\mathrm{Bi} + 2}{\mathrm{Bi}} - \xi^2 \right) \begin{cases} 1 - \exp [- p_1^{(1)} (\mathrm{Bi}, m) \mathrm{Fo}] & X > \mathrm{Fo} \\ 1 - \exp [- p_1^{(1)} (\mathrm{Bi}, m) X] & X < \mathrm{Fo} \end{cases} . \qquad (26)$$

If it is necessary to refine solutions, in the subsequent approximations the relative temperature without a heat source is reduced to the form

$$\Theta_n (\xi, X, \mathrm{Fo}, \mathrm{Bi}, m) = \sum_{i=1}^{n} \psi_i^{(n)} (\xi, \mathrm{Bi}, m) \exp [- p_i^{(n)} (\mathrm{Bi}, m) f] ,$$

where f = Fo for $X > \mathrm{Fo}$ and $f = X$ for $X < \mathrm{Fo}$. The functions $\psi_i^{(n)}$ and the eigenvalues $p_i^{(n)}$ should be found from the determining system composed for fixed Bi and m.

The local heat flux over the round-tube surface and the mean-integral temperature in the third approximation with boundary conditions of the first kind are equal to

$$\tilde{q} (X, \mathrm{Fo}) = \frac{q (X, \mathrm{Fo}) R}{\lambda (T_0 - T_\mathrm{w})} = - \left(\frac{\partial \Theta}{\partial \xi} \right)_{\xi=1} = 2.022 \exp (- 5.783 f) + 1.654 \exp (- 30.712 f) +$$

$$+ 11.346 \exp (- 113.503 f) ,$$

$$\langle \Theta (X, \mathrm{Fo}) \rangle = 2 \int_0^1 \Theta \xi d\xi = 0.692 \exp (- 5.783 f) + 0.136 \exp (- 30.712 f) + 0.110 \exp (- 113.503 f) .$$

Problems of convective heat transfer in tubes and channels with piston flows can be considered as problems of heat transfer with uniform motion of rods without channel walls and can serve as mathematical models for investigation of the dynamics of cooling of hot-rolled stock of round rods and plane metal or polymer sheets. In the solutions found with f = Fo, the process is described in that part of a body which at the moment Fo = 0 was already in the zone $X > 0$, while with $f = X$ for a body which is pulled with velocity

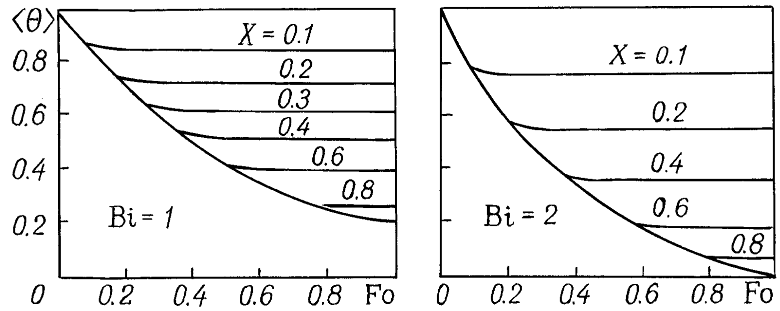

Fig. 2. Change in the mean-integral unsteady temperature in a rod moving with velocity w_0 for Bi = 1 and Bi = 2 at particular fixed points X.

w_0 from the zone $X = 0$. The mean-integral temperature from the first term of solution (26) for a round rod cooled by convection and radiation is reduced to the form

$$\langle \Theta (X, \mathrm{Fo}, \mathrm{Bi}) \rangle = \frac{\langle T (X, \mathrm{Fo}) \rangle - (T_\mathrm{w} - q/\alpha)}{T_0 - (T_\mathrm{w} - q/\alpha)} = N (\mathrm{Bi}) \begin{Bmatrix} \exp [- p_1^{(1)} (\mathrm{Bi}, 1)\, \mathrm{Fo}]\,, \ X > \mathrm{Fo} \\ \exp [- p^{(1)} (\mathrm{Bi}, 1)\, X]\,, \ X < \mathrm{Fo} \end{Bmatrix},$$

$$N = \frac{3\, (\mathrm{Bi}^2 + 8\, \mathrm{Bi} + 16)}{4\, (\mathrm{Bi}^2 + 6\, \mathrm{Bi} + 12)}\,. \tag{27}$$

For small Bi numbers ($\mathrm{Bi}^2 \approx 0$) $N(\mathrm{Bi}) = 3(8\mathrm{Bi} + 16)/4(6\mathrm{Bi} + 12) = 1$ and $N(0.5) = 60/61 = 0.984$, $N(1) = 75/76 = 0.987$, $N(2) = 0.964$. The unsteady abrupt decrease in the mean-integral temperature (heat content) at different points X in convective and conductive cooling of a rod (fittings, wire, thread) pulled from the point $X = 0$ ($z = 0$) with a constant velocity w_0 calculated from formula (27) for Bi = 1 and Bi = 2 is shown in Fig. 2.

In the case of cooling of a rod (a slab) by convection ($T_0 >> T_\mathrm{w}$) and radiation with incident intensity in the form $q(X) = q \exp (- \mathrm{Pd}\, X)$ a change in the temperature is determined by the formula

$$\Theta (\xi, X, \mathrm{Bi}, m) = \frac{T - (T_\mathrm{w} - q/\alpha \exp (- \mathrm{Pd}\, X))}{T_0 - T_\mathrm{w}} = \frac{p_1^{(1)} (\mathrm{Bi}, m)}{2\, (m + 1)} \left(\frac{\mathrm{Bi} + 2}{\mathrm{Bi}} - \xi^2 \right) \exp [- p_1^{(1)} (\mathrm{Bi}, m)\, X] +$$

$$+ \frac{p_1^{(1)} (\mathrm{Bi}, m)\, q/\alpha}{2\, (m + 1)\, (T_0 - T_\mathrm{w})} \left(\frac{\mathrm{Bi} + 2}{\mathrm{Bi}} - \xi^2 \right) \left[\exp (- \mathrm{Pd}\, X) + \frac{p_1^{(1)} (\mathrm{Bi}, m)\, (\exp (- \mathrm{Pd}\, X) - \exp (- p_1^{(1)} X))}{\mathrm{Pd} - p_1^{(1)} (\mathrm{Bi}, m)} \right].$$

In the suggested numerical-analysis method, the velocity function is used to calculate the coefficients B_{jk} and F_j. It is subjected only to integration and is not differentiated anywhere; therefore, a small difference in the velocity profiles leads to insignificant deviations in intermediate calculations and determinations of solutions as a whole, as was the case in the temperature representations (18)–(23). Here, the continuous dependence of calculation of the matrix elements $\|B\|$ on the type of velocity is unambiguously manifested in calculations of the eigenvalues $s_i^{(n)}$, $i = \overline{1, n}$, which determine the rate of stabilization of the thermoinertial systems along the fluid flow. Stability of the algorithm to an error of the velocity-distribution function makes it possible to implement the method by calculating the coefficients B_{jk} and F_j using numerical integration in the cases where the velocity values are known at discrete points as a result of numerical solution of the Poisson equation or are found by some other method. The multipurpose character and the efficiency of the method lie in the fact that it can be implemented for any prescribed values of the stabilized velocity, including the velocities in turbulent modes of flows.

Let us consider the problem of heat transfer (mass transfer) in a two-dimensional running fluid layer with height h ($0 \le \xi = x/h \le 1$) when the steady-state velocity of flow is equal to

$$w(\xi) = W(\xi)/w_0 = 1.5\,(2\xi - \xi^2)\,, \quad w_0 = gh^2 \sin\beta/3\nu\,, \quad \max W = W(1) = 1.5 w_0\,,$$

where β is the slope of the plane of running down toward the horizon, g is the free-fall acceleration, and ν is the kinematic viscosity, and on the upper free surface the boundary conditions of the third kind are prescribed and the lower wall is assumed to be adiabatic (impermeable), i.e., conditions similar to (8) are prescribed. The procedure of implementation of the method with constant boundary loadings ($\varphi(X, \text{Fo}) = T_w$, $\varphi_0(\text{Fo}) = T_0$, $q(X, \text{Fo}) = q = \text{const}$, $f(X, \text{Fo}) = 0$) leads to the representation of the solution in the first approximation

$$\Theta(\xi, X, \text{Fo}, \text{Bi}) = \frac{T - (T_w + q/\alpha)}{T_0 - (T_w + q/\alpha)} = \left(\frac{\text{Bi}+2}{\text{Bi}} - \xi^2\right)\left\{\begin{matrix} M(\text{Bi}) \exp[-p_1^{(1)}(\text{Bi})\,\text{Fo}] & \text{for } X > \gamma\,\text{Fo} \\ N(\text{Bi}) \exp[-s_1^{(1)}(\text{Bi})\,X] & \text{for } X < \gamma\,\text{Fo} \end{matrix}\right\}, \qquad (28)$$

where

$$p_1^{(1)}(\text{Bi}) = \frac{5\,\text{Bi}\,(\text{Bi}+3)}{2\,\text{Bi}^2 + 10\,\text{Bi} + 15}\,; \quad s_1^{(1)}(\text{Bi}) = \frac{280\,\text{Bi}\,(\text{Bi}+3)}{3\,(27\,\text{Bi}^2 + 154\,\text{Bi} + 280)}\,; \quad M(\text{Bi}) = 0.5 p_1^{(1)}(\text{Bi})\,;$$

$$N(\text{Bi}) = \frac{7\,\text{Bi}\,(11\,\text{Bi}+10)}{2\,(27\,\text{Bi}^2 + 154\,\text{Bi} + 280)}\,; \quad \gamma(\text{Bi}) = \frac{p_1^{(1)}}{s_1^{(1)}} = \frac{3\,(27\,\text{Bi}^2 + 154\,\text{Bi} + 280)}{56\,(2\,\text{Bi}^2 + 10\,\text{Bi} + 15)}\,.$$

The largest error of this solution is attained when $\text{Bi} = \infty$, and with the boundary conditions of the first kind we have

$$\Theta(\xi, X, \text{Fo}) = \frac{T(\xi, X, \text{Fo}) - T_w}{T_0 - T_w} = (1 - \xi^2)\left\{\begin{matrix} 1.25 \exp(-2.5\,\text{Fo}) & \text{for } X > 0.723\,\text{Fo} \\ 1.426 \exp(-3.457\,X) & \text{for } X < 0.723\,\text{Fo} \end{matrix}\right\}. \qquad (29)$$

Refinements of the lower main lines in the solutions (28) and (29) at steady-state heat loadings are reduced to solving the system

$$\sum_{k=1}^{n} a_k^*(s)\,(A_{jk} + B_{jk}\,s) = [T_0 - (T_w + q/\alpha)]\,F_j\,, \quad j = \overline{1, n}\,;$$

therefore we provide the recurrence formulas of calculation of the coefficients:

$$A_{jk} = -\int_0^1 \frac{d^2\psi_k}{d\xi^2}\,\psi_j\,d\xi = 4kj\left(\frac{1}{2j + 2k - 1} + \frac{1}{\text{Bi}}\right) = A_{kj} > 0\,; \quad B_{jk} = 1.5\int_0^1 (2\xi - \xi^2)\,\psi_j\,\psi_k = b_k b_j +$$

$$+ \frac{3\,(k + j + 2)}{2\,(k + j + 1)\,(2k + 2j + 3)} - \frac{3}{2}\left[\frac{b_j\,(k+2)}{(k+1)\,(2j+1)} + \frac{b_k\,(j+2)}{(j+1)\,(2k+1)}\right]; \quad F_j = 1.5\int_0^1 (2\xi - \xi^2)\,\psi_j\,d\xi =$$

$$= b_j - \frac{3\,(j+2)}{(2j+2)\,(2j+3)}\,; \quad b_j = \frac{\text{Bi}+2}{\text{Bi}}\,. \qquad (30)$$

TABLE 1. Calculation of the Relative Concentration in the Second and Third Approximations

n	i	$\psi_i^{(n)}(\xi)$	$s_i^{(n)}$
2	1	$1.4018 - 1.2516\xi^2 - 0.1502\xi^4$	3.4477
	2	$-0.6607 + 3.6009\xi^2 - 2.9302\xi^4$	27.9894
3	1	$1.3487 - 0.7542\xi^2 - 1.2692\xi^4 + 0.6747\xi^6$	3.4185
	2	$-0.6412 + 3.9861\xi^2 - 4.1493\xi^4 + 0.8044\xi^6$	27.7298
	3	$0.5048 - 6.5677\xi^2 + 15.7484\xi^4 - 9.6855\xi^6$	85.4415

In chemical technology, use is made of apparatuses in which a thin running-down film absorbs (desorbs) a component from the upper surrounding medium, which is dissolved in a fluid. Let the sought concentration be equal to $U(\xi, X)$ (instead of the temperature $T(\xi, X)$); then $X = \frac{1}{\mathrm{Pe}}\frac{z}{h}$ and $\mathrm{Pe} = \frac{w_0 h}{a_\mathrm{m}}$. The relative concentration $\tilde{\theta}(\xi, X)$ with boundary conditions $[U(\xi, X)]_{X=0} = U_0$, $[U(\xi, X)]_{\xi=1} = U_\mathrm{w}$, $\left(\frac{\partial U}{\partial \xi}\right)_{\xi=0} = 0$ by composing the determining systems with the aid of the coefficients (30) for Bi = ∞ and subsequent implementation of the algorithm is represented by the expression

$$\theta_n(\xi, X) = \sum_{i=1}^{n} \psi_i^{(n)}(\xi) \exp(-s_i^{(n)} X)\,, \tag{31}$$

where the eigenfunctions $\psi_i^{(n)}(\xi)$ and the numbers $s_i^{(n)}$ as the results of refinement of the lower line of solution (29) are given in Table 1.

The mass-mean, with respect to the thickness of the thin film, concentration in the first approximation is written as

$$\langle \theta_3(X) \rangle = 1.5 \int_0^1 \theta_3(\xi, X)(2\xi - \xi^2)\, d\xi = 0.788 \exp(-3.419X) + 0.134 \exp(-27.730X) +$$
$$+ 0.031 \exp(-85.442X)\,. \tag{32}$$

An investigation of this problem in a rigorous nonalternative space leads to determination of the coordinate functions by solving the Sturm–Liouville problem, whose eigenfunctions are expressed in terms of the hypergeometric Gaussian series in the form [8]

$$Y_i(\xi) = B\gamma_i^{1/2}(\xi - 1) \exp\left[-\frac{\gamma_i(\xi - 1)^2}{2}\right] F\left(\frac{3}{4} - \frac{\gamma_i}{4};\ \frac{3}{4};\ \gamma_i(\xi - 1)^2\right),$$

where the values of γ_i are the roots of the characteristic equation

$$\gamma(3 - \gamma) F\left(\frac{3}{4} - \frac{\gamma}{4};\ \frac{5}{2};\ \gamma\right) - 3(\gamma - 1) F\left(\frac{3}{4} - \frac{\gamma}{4};\ \frac{3}{2};\ \gamma\right) = 0\,,$$

In this work, the values $\gamma_1 = 2.2631$, $\gamma_2 = 6.2977$, and $\gamma_3 = 10.3077$ are found and the solution is given in the form

$$\theta(\xi, X) = \sum_{i=1}^{\infty} A_i Y_i(\xi) \exp\left(-\frac{2}{3}\gamma_i^2 X\right). \tag{33}$$

According to representation (31), a sequence of numbers $s_1^{(1)}$, $s_1^{(2)}$, ..., $s_1^{(n)}$ must converge to the exact eigenvalue $s_1 = 2/3\gamma_1^2 = 3.4144$. The value $s_1^{(1)} = 3.4568$ exceeds this value by 1.24%, $s_1^{(2)} = 3.4477$ exceeds it by 0.95%, and $s_1^{(3)} = 3.4185$ exceeds the exact value by only 0.12%. In solution (31), the poorer convergences of the eigenfunction $\psi_n^{(m)}(\xi)$, the number $s_n^{(n)}$, $m \geq n - 1$, and the last term as a whole $\psi_n^{(n)}(\xi) \exp(-s_n^{(n)} X)$ to the value $A_n \gamma_n(\xi) \exp\left(-\frac{2}{3}\gamma_n^2 X\right)$ can be assigned to the higher efficiency of the suggested method in the following way. In solution (31), the quantity $\psi_n^{(n)} \exp(-s_n^{(n)} X)$ contains, in addition to the value of its own spectrum of decomposition, the influence of the remainder, which is not taken into account in calculation by the corresponding partial sum of the solution (33). Therefore, the solutions (31) and (32) in the third approximation, for instance, describe the process better than by the partial sum of third order of the exact solution.

In conclusion, it should be noted that the suggested numerical-analysis method of calculation by interpolation of the experimentally measured temperature of the surface of a thermally thin channel wall by the function $\varphi(\xi, X, \mathrm{Fo})$, which is a result of the partial response at the boundary of conjugate heat transfer between the flowing fluid inside the channel and the external flow of the medium propagating along the ξ-axis, allows determination of the temperature field in the channel, which makes it possible to investigate the regularities of changes in the local heat flux and heat content with time and along the fluid flow by performing differentiation and integration with respect to the elliptic coordinates ξ, η of the found solutions. Thus, it becomes unnecessary to calculate the local and integral Nusselt numbers. In the cases where the temperature of the transversely incident fluid flow is higher or lower than the fluid temperature in the channel, the rate of exponential stabilizations of heat transfer under real conditions will be higher or lower, respectively, than in the calculations of heat transfer in channels with isothermal flow.

If we set $w(\xi, \eta) = 0$ on the left-hand side of Eq. (1), we will obtain the heat-conduction equation for long rods. Then the interpolation of the function of two variables $\varphi(\xi, \mathrm{Fo})$ as a change in the temperature on the surface of a solid prism or a body of revolution around the ξ-axis ($\eta^2 \to \eta^2 + \xi^2$) allows one to simplify the solution of the problems of conjugate heat transfer between a solid and an incident flow of a medium.

NOTATION

w_0, mean integral velocity; a, thermal diffusivity; μ, dynamic viscosity; μ_w and μ_0, viscosity in the near-wall layer and in the core of the fluid flow, respectively; h and b, lengths of the intervals of change of the running coordinates x, y $(0 \leq x \leq h, -b \leq y \leq b)$ in the region D of the channel regions; z, unilateral coordinate $(0 \leq z \leq \infty)$ along the channel; $\xi = x/h$, $\eta = y/b$, $X = z/h\mathrm{Pe}$, relative coordinates of the running point; t, time; Fo, Pe, and Bi, Fourier, Péclet, and Biot numbers related to h; W and w, dimensional and relative local velocities of the fluid flow $(w = W/w_0)$; $\Theta = (T - T_0)/(T_w - T_0)$, relative excess temperature in the channel; $T(\xi, \eta, X, \mathrm{Fo})$, temperature at the running point $M(\xi, \eta, X)$ at the time Fo; T_w, temperature of the external medium or the channel wall; m, parameter in the combined equation of heat transfer for a plane channel ($m = 0$) and a round tube ($m = 1$); β, parameter of the geometric shape of the channel ($\beta = h^2/b^2$); δ, correcting parameter of the distribution nonuniformity of the reciprocal local viscosity $k = \mu^{-1}$ in heating ($\delta > 0$) and cooling ($\delta < 0$) of the fluid and $\delta = 0$ in isothermal flow; F, hypergeometric Gaussian series; T_0, initial temperature distribution or temperature at the channel inlet; $\|.\|$, $|.|$, notation of the matrices and the determinants, respectively; $\tilde{\theta} = (u - u_w)/(u_0 - u_w)$, relative mass concentration; a_m, mass-transfer coefficient; $\widetilde{Pe} = W_0 h / a_m$, Péclet number for mass transfer; s, p, parameters of double Laplace–Carson transformation; $\doteqdot$, sign of one-to-one correspondence between the inverse transform and the transform; $T^*(\xi, s)$, Laplace transform of the temperature $T(\xi, X)$ with respect to X; q, radiative flux ($q > 0$) or radiation ($q < 0$) in the generalized boundary conditions of the third kind; $\langle \Theta(X, \text{Fo}) \rangle$, mass-mean relative temperature over the channel cross section; $-s_i^{(n)}$, $-p_i^{(n)}$ ($i = \overline{1, n}$), simple roots of the algebraic equations $|A + sB| = 0$ and $|A + pC| = 0$; n, order of approximation.

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